MAT-17441 : optimisation — oct. 2005

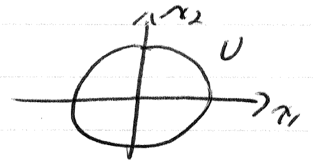
Solutionnaire, examen 2

1. [20 pts]

a) Si $\bar{x} \in \operatorname{int} U \Rightarrow$ Cond. d'opt est $\nabla f(\bar{x}) = 0$ 2/

5/ Or $\nabla f = a \neq 0$ 2/ $\Rightarrow$ impossible 1/

b)



Ex: $\min_{x \in U} x_1^2 + x_2^2$ 3/

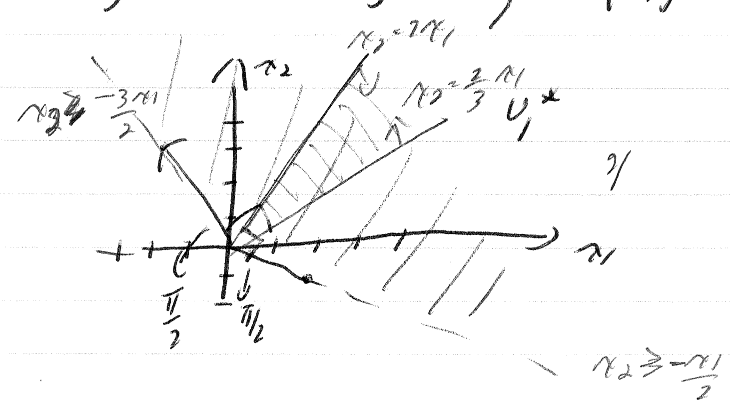
5/ de minimum $(0,0) \in \operatorname{int} U$ 2/

c) $x_2 \le 2x_1 \iff 2x_1 - x_2 \ge 0$

5/ $3x_2 \ge 2x_1 \iff 3x_2 - 2x_1 \ge 0$

$U_1 = \{x \mid Bx \ge 0\}$ avec $B = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}$ 1/

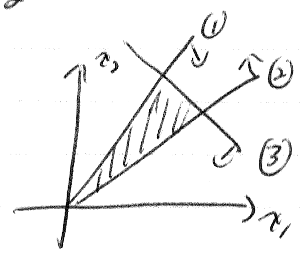
$U_1^* = \{B^T \lambda \mid \lambda \ge 0\} = \left\{ \lambda_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 3 \end{pmatrix} \;\middle|\; \lambda_i \ge 0 \right\}$ 2/



2/

5/

d) $U_2 =$



①: $x_2 \le 2x_1$

② $x_2 \geq \frac{2}{3} x_1$

③ $x_1 + x_2 \le 3$

2/

3/

$U_2^* = U_1^*$

2. [20 pts]

a) [12 pts]

$\max \ (x, y)$
$\sum x_i^2 = 1$
$\sum y_i^2 = 1$

$$L(x, y, \lambda_1, \lambda_2) = (x, y) + \lambda_1 \left(\sum x_i^2 - 1\right) + \lambda_2 \left(\sum y_i^2 - 1\right)$$

① $\frac{\partial L}{\partial x} = y + 2\lambda_1 x = 0$

② $\frac{\partial L}{\partial y} = x + 2\lambda_2 y = 0$

4/

③ $\frac{\partial L}{\partial \lambda_1} = \sum x_i^2 - 1 = 0$

④ $\frac{\partial L}{\partial \lambda_2} = \sum y_i^2 - 1 = 0$

De ① $\Rightarrow \|y\| = 2|\lambda_1| \, \|x\| \Rightarrow |\lambda_1| = 1/2$

$\Rightarrow \lambda_1 = \pm 1/2$

/6

Si: $\lambda_1 = 1/2 \Rightarrow y = -x$

$\Rightarrow$ ② $x - 2\lambda_2 x = 0 \Rightarrow (1 - 2\lambda_2) x = 0 \Rightarrow \lambda_2 = 1/2$

et $f(x, y) = -1$ min $x \neq 0$

Si $\lambda_1 = -1/2 \Rightarrow y = x$

$\Rightarrow (2) \quad x + 2\lambda_2 x = 0 \Rightarrow 1 + 2\lambda_2 = 0 \Rightarrow \lambda_2 = -1/2$

$\Rightarrow f(x,y) = 1 \Rightarrow$ max

So $x = y \Rightarrow$ max

b) [6 pts]

On a que $\left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right) \leq 1$

$\Rightarrow (x,y) \leq \|x\| \, \|y\|$

c) On a égalité si
[3 pts]

$\left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right) = 1$

$\Rightarrow \frac{x}{\|x\|} = \frac{y}{\|y\|} \Rightarrow x = \lambda y \quad \lambda > 0$

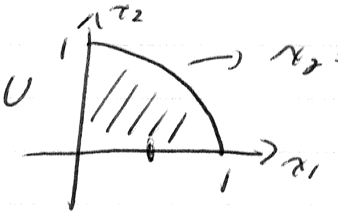
selon a)

(4)

3. [20 pts]

a) [5 pts]

2/



2

b) [4 pts] $f(x_1,x_2) = \frac{1}{2}\left(4x_1^2 + 2x_2^2\right) - 2x_1 + 4x_2$

4/

$= \frac{1}{2}(Ax, x) - (b, x)$

avec $A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} > 0$

$\therefore$ f strict. convexe 2

U est borné, fermé

$\Rightarrow \exists!$ solution 2/

c) On pose: $g_1 = -x_1 \leq 0$; $g_2 = -x_2 \leq 0$ et $g_3 = x_2 + x_1^2 - 1 \leq 0$

[4 pts] KKT:

$$\begin{cases} 4x_1 - 2 - \lambda_1 + 2x_1\lambda_3 = 0 \\ 2x_2 + 4 - \lambda_2 + \lambda_3 = 0 \\ \lambda_1, \lambda_2, \lambda_3 \geq 0 \\ \lambda_1 x_1 = 0 \\ \lambda_2 x_2 = 0 \\ \lambda_3 (x_2 + x_1^2 - 1) = 0 \end{cases}$$

-1 KKT

1

Si $\lambda_1 \neq 0$ et $\lambda_2 \neq 0$ $\Rightarrow$ $x_1 = x_2 = 0$ $\Rightarrow$ $-2 - \lambda_1 = 0$ #

donc $\lambda_1 = 0$ ou $\lambda_2 = 0$

4/ calcul

$\hookrightarrow$ Si $\lambda_1 \neq 0$ $\Rightarrow x_1 = 0$ $\Rightarrow$ $-2 - \lambda_1 = 0$ #

$\Rightarrow \boxed{\lambda_1 = 0}$

Si $\lambda_2 = 0$ $\Rightarrow$ $2x_2 + 4 + \lambda_3 = 0 \Rightarrow \lambda_3 = -(2x_2 + 4) \leq 0$ #

$\Rightarrow$ $\lambda_2 \neq 0$ $\Rightarrow$ $\boxed{x_2 = 0}$

Le système se réduit à :

$$\begin{cases} 4x_1 - 2 + 2x_1\lambda_3 = 0 \\ 4 - \lambda_2 + \lambda_3 = 0 \\ \lambda_3 (x_1^2 - 1) = 0 \end{cases}$$

Si $\lambda_3 = 0 \Rightarrow 4x_1 = 2 \Rightarrow x_1 = 1/2$

$\lambda_2 = 4$

$\Rightarrow x_1 = 1/2 \quad \lambda_1 = 0 \quad \lambda_3 = 0$

$x_2 = 0 \quad \lambda_2 = 4$

5

Note: si $\lambda_3 \neq 0 \Rightarrow x_1 = \pm 1$ impossible

$x_1 = 1 \Rightarrow 4 - 2 + 2\lambda_3 = 0 \Rightarrow \lambda_3 < 0$ #

$x_1 = -1 < 0 \Rightarrow \cancel{-4 - 2 - 2\lambda_3 = 0 \Rightarrow \lambda_3 < 0}$ #

4. [20 pts]

a) [12 pts]

KKT :

$$2(x_1 - 3) + 2\lambda_1 x_1 = 0$$
$$2(x_2 - 4) + 2\lambda_1 x_2 = 0$$
$$\lambda_1 \geq 0 \; , \quad x_1^2 + x_2^2 \leq 1$$
$$\lambda_1 (x_1^2 + x_2^2 - 1) = 0$$

4

La solution (x_1, x_2) correspond au pt. de projection :

$$\begin{cases} x_1^2 + x_2^2 = 1 \\ x_2 = \frac{4}{3} x_1 \end{cases} \Rightarrow x_1^2 \left(1 + \frac{16}{9}\right) = 1$$

6

$\Rightarrow x_1 = 3/5$ et $x_2 = 4/5$

Vérification de KKT : $(1 + \lambda_1) x_1 = 3 \Leftarrow 1 + \lambda_1 = 3/(3/5) = 5$

$\Rightarrow \lambda_1 = 4 \geq 0$

2

De même : $x_2(1+\lambda_1) = 4$

$1+\lambda_1 = 4/x_2 = 4/\tfrac{4}{5} = 5 \Rightarrow \lambda_1 = 4 \geq 0$

b) [6 pts]

$$L(x_1,x_2,x_3,\lambda) = (x_1-3)^2 + (x_2-4)^2 + \lambda_1(x_1^2+x_2^2+x_3^2-1)$$

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = 2(x_1-3) + 2\lambda_1 x_1 = 0 \\ \dfrac{\partial L}{\partial x_2} = 2(x_2-4) + 2\lambda_1 x_2 = 0 \\ \dfrac{\partial L}{\partial x_3} = 2\lambda_1 x_3 = 0 \\ \dfrac{\partial L}{\partial \lambda} = 0 \end{cases}$$ 3/

Si $\lambda_1 = 0 \Rightarrow x_1 = 3$ et $x_2 = 4$ $\cancel{\times}$

$\Rightarrow \lambda_1 \neq 0 \Rightarrow x_3 = 0 \Rightarrow x_1^2 + x_2^2 = 1$ comme en a)

$\therefore$ $x_1 = 3/5$
$x_2 = 4/5$
$x_3 = 0$ 3/

c) [2 pts]

idem

(7)

5. [20 pts]

a) [4 pts] $L(x,\lambda) = \frac{1}{2}(Ax,x) - (b,x) + (\lambda, Bx)$

b) [6 pts]

$$\min_x L(x,\lambda)$$

$$\nabla_x L = Ax - b + B^t\lambda = 0 \qquad 4$$

$$\Rightarrow Ax = b - B^t\lambda$$

$$\Rightarrow x = A^{-1}(b - B^t\lambda) \qquad 2$$

c) [6 pts]

$$-L(x,\lambda) = -\frac{1}{2}(Ax,x) + (b,x) - (\lambda, Bx)$$

$$= -\frac{1}{2}(b - B^t\lambda, x) + (b,x) - (\lambda, Bx)$$

$$= -\frac{1}{2}(b,x) + (b,x) + \frac{1}{2}(\lambda, Bx) - (\lambda, Bx)$$

$$= \frac{(b,x)}{2} - \frac{(\lambda, Bx)}{2} = -\frac{(B^t\lambda, x)}{2} + \frac{(b,x)}{2}$$

$$= -\frac{1}{2}(B^t\lambda, A^{-1}(b - B^t\lambda)) + \frac{1}{2}(b, A^{-1}(b - B^t\lambda))$$

$$= \frac{1}{2}(B^t\lambda, A^{-1}B^t\lambda) - \frac{1}{2}(B^t\lambda, A^{-1}b) - \frac{1}{2}(b, A^{-1}B^t\lambda) + \frac{1}{2}(b, A^{-1}b)$$

$$h(\lambda) = \frac{1}{2}(BA^{-1}B^t\lambda, \lambda) - (BA^{-1}b, \lambda) + \frac{1}{2}(b, A^{-1}b)$$

d) [4 pts] $C = BA^{-1}B^t$ sym > 0 et $d = BA^{-1}b$

$e = \frac{1}{2}(b, A^{-1}b)$