MAT-17441 : optimisation

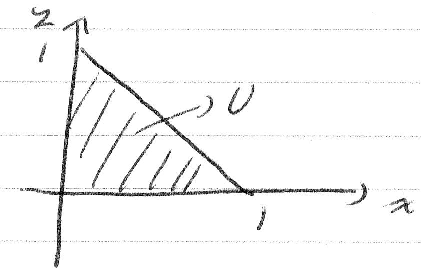
Solutionnaire, examen 2, automne 2004

1. [20 pts]

a) [5 pts] Il s'agit de résoudre

$$\min_{(x,y)\in U} x^2 - 2x - y$$

où U correspond à



$$g_1(x,y) = x+y-1 \le 0$$
$$g_2(x,y) = -x \le 0$$
$$g_3(x,y) = -y \le 0$$

$$Df = (2x-2, -1)$$

$$\Rightarrow \quad Dg_1 = (1,1)$$
$$Dg_2 = (-1,0)$$
$$Dg_3 = (0,-1)$$

Les conditions d'optimalité s'écrivent:

$$Df + \lambda_1 Dg_1 + \lambda_2 Dg_2 + \lambda_3 Dg_3 = 0$$

$$\begin{cases} 2x - 2 + \lambda_1 - \lambda_2 = 0 \\ -1 + \lambda_1 - \lambda_3 = 0 \\ \lambda_1, \lambda_2, \lambda_3 \ge 0 \\ x+y-1 \le 0, \quad x \ge 0, y \ge 0 \\ \lambda_1 (x+y-1) = 0 \\ \lambda_2 x = 0 \\ \lambda_3 y = 0 \end{cases}$$

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2/

b) [10 pts]

Si $\lambda_1 = 0 \Rightarrow \lambda_3 = -1$ #

$\Downarrow$

$\boxed{\lambda_1 > 0} \Rightarrow \boxed{x + y - 1 = 0}$

Si $\lambda_2 \neq 0 \Rightarrow x = 0 \Rightarrow y = 1 \Rightarrow \lambda_1 - \lambda_2 = 2$

$\lambda_1 - \lambda_3 = 1$

$\Downarrow$ $\lambda_3 = 0$ $\Rightarrow$ $\lambda_1 = 1$, $\lambda_2 = -1$ #

$\Downarrow$

$\boxed{\lambda_2 = 0}$

$$\Rightarrow \quad \begin{aligned} 2x - 2 + \lambda_1 &= 0 \\ -1 + \lambda_1 - \lambda_3 &= 0 \\ x + y &= 1 \\ \lambda_1 &> 0 \end{aligned}$$

Si $\lambda_3 \neq 0 \Rightarrow y = 0 \Rightarrow x = 1$ ~~...~~ $\lambda_1 = 0$ #

$\Downarrow$

$\boxed{\lambda_3 = 0} \Rightarrow \lambda_1 = 1$

$2x - 2 + 1 = 0 \Rightarrow x = 1/2 \Rightarrow y = 1/2$

Eo $\quad (x, y) = (1/2, 1/2)$

$\lambda_1 = 1$

$\lambda_2 = 0$

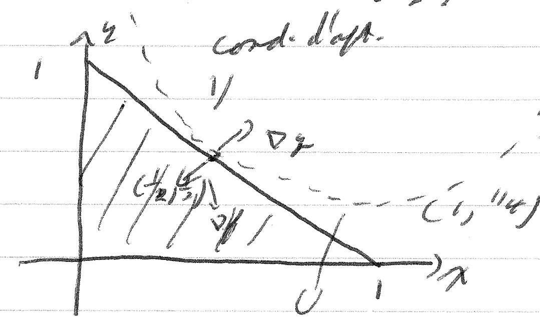
$\lambda_3 = 0$

−3 pts si autre cas!
ou −2

c) [5 pts] $\quad \nabla f(\tfrac{1}{2},\tfrac{1}{2}) = (-1, -1) \quad \nabla g_1(\tfrac{1}{2},\tfrac{1}{2}) = (1,1)$

Fig. U 2 pts

2 pts forme U



2 pts courbe de niveau

$f = x^2 - 2x - y$

$f(\tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{1}{4} - 1 - \tfrac{1}{2} = -\tfrac{5}{4}$

$x^2 - 2x - y = -5/4$ courbe de niveau

$y = (x-1)^2 - 1 + 5/4$

$y = (x-1)^2 + 1/4$

(4)

Si $\quad 2-2x+\lambda_1-\lambda_2=0$

$\quad 1+\lambda_1-\lambda_3=0$

$\lambda_1(x+y-1)=0$

$\lambda_2 x=0$

$\lambda_3 y=0$

Si $\lambda_1=0 \quad \Rightarrow \lambda_3=1 \Rightarrow y=0 \Rightarrow$

$2-2x-\lambda_2=0$

$\cancel{\lambda_1(x+y-1)=0}$

$\lambda_2 x=0$

Si $\lambda_2=0 \Rightarrow x=1 \Rightarrow$ $x=1$, $y=0$, $\lambda_1=0$, $\lambda_2=0$, $\lambda_3=1$ $\quad f(1,0)=1$ ✓

Si $\lambda_2\neq 0 \Rightarrow x=0 \Rightarrow 2-\lambda_2=0 \Rightarrow \lambda_2=2$

MIN ← $x=0$, $y=0$, $\lambda_1=0$, $\lambda_2=2$, $\lambda_3=1$ $\quad f(0,0)=0$ ✓

Si $\lambda_1\neq 0 \Rightarrow x+y-1=0$

$2-2x+\lambda_1-\lambda_2=0$

$1+\lambda_1-\lambda_3=0$

Si $\lambda_3=0 \Rightarrow \lambda_1=-1$ ✗

$\lambda_3\neq 0 \quad y=0 \Rightarrow x=1 \quad (1,0)$

$\Downarrow$

$\lambda_2=0 \Rightarrow \lambda_1=0$ #

Q 2 [20 pts]

a) [6 pts]

$$\min \cancel{f} \underbrace{(x^2+y^2+z^2)}_{f(x,y,z)}$$

$$\begin{cases} x+y+2z = 2 \\ z = x^2+y^2 \end{cases}$$

$g_1(x,y,z) = x+y+2z-2$ 2/

$g_2(x,y,z) = x^2+y^2-z = 0$ 2/

$f(x,y,z) = x^2+y^2+z^2$ 2/

b) [14 pts]

$$L(x,y,z,\lambda_1,\lambda_2) = x^2+y^2+z^2+\lambda_1(x+y+2z-2)+\lambda_2(x^2+y^2-z)$$

Les cond. d'opt. s'écrivent:

$$\begin{cases} 2x + \lambda_1 + 2\lambda_2 x = 0 \\ 2y + \lambda_1 + 2\lambda_2 y = 0 \\ 2z + 2\lambda_1 - \lambda_2 = 0 \\ x^2+y^2-z=0 \\ x+y+2z-2=0 \end{cases}$$

4/

① - ② $\Rightarrow$ $2(x-y) + 2\lambda_2(x-y) = 0$

$\Leftrightarrow$ $(x-y)(1+\lambda_2) = 0$

1e cas: $\cancel{\lambda_2} = -1$

2/ $\Rightarrow 2x + \lambda_1 - 2x = 0 \Rightarrow \lambda_1 = 0$

$2y + \lambda_1 - 2y = 0$

$2z + 2\lambda_1 + 1 = 0 \Rightarrow z = -1/2$

$\Rightarrow x^2 + y^2 = z = -1/2$ impossible

2e cas: $x = y$

dans les 2 contraintes:

4/ $z = 2x^2$

$2x + 2z - 2 = 0 \Leftrightarrow x + z = 1$

$1 - x = 2x^2 \Rightarrow \boxed{x = -1} \Rightarrow (-1, -1, 2)$

$\Rightarrow x = 1/2 \Rightarrow \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

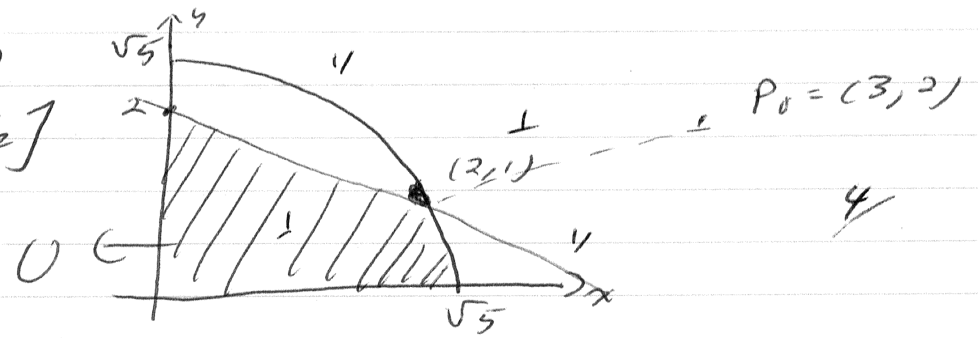
(x,y,z)	f	
$(-1,-1,2)$	6	max
$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$	$3/4$	min

le plus proche: $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ min de distance $= \frac{\sqrt{3}}{2}$

le plus éloigné: $(-1,-1,2)$ max le distance $= \sqrt{6}$

Q 3 [25 pts]

a) [4 pts]



b) [5 pts]

$f = \dfrac{(x-3)^2 + (y-2)^2}{2}$ est coercive strict. convexe 2/

U est borné et fermé 1/

$\Rightarrow \min_{(x,y)\in U} f(x,y)$ admet une sol. unique. 2/

c) [6 pts] Les conditions d'optimalité s'écrivent

$$\begin{cases} x-3 + 2\lambda_1 x + \lambda_2 - \lambda_3 = 0 \\ y-2 + 2\lambda_1 y + 2\lambda_2 - \lambda_4 = 0 \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \\ x^2+y^2-5 \leq 0, \quad x+2y-4 \leq 0, \quad x \geq 0, \quad y \geq 0 \\ \lambda_1 (x^2+y^2-5) = 0 \\ \lambda_2 (x+2y-4) = 0 \\ \lambda_3 x = 0 \end{cases}$$

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(6)

d) [10 pts]

Géométriquement, il est clair que la solution doit être le pt $(2,1)$. Vérifions les cond. de KKT en ce pt.

$(x,y) = (2,1) \Rightarrow \lambda_3 = 0 = \lambda_4$

dans les 2 ième éq. de KKT $\Rightarrow$

Vérification

5/

$$-1 + 4\lambda_1 + \lambda_2 = 0 \iff 4\lambda_1 + \lambda_2 = 1$$
$$-1 + 2\lambda_1 + 2\lambda_2 = 0 \qquad 2\lambda_1 + 2\lambda_2 = 1$$

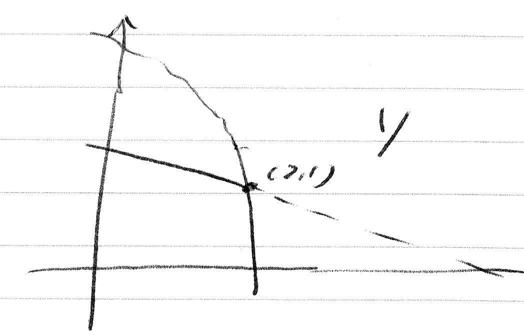
de solution $\lambda_1 = 1/6$

$\lambda_2 = 1/3$

ô

4/

$$\begin{cases} x = 2 \\ y = 1 \\ \lambda_1 = 1/6 \\ \lambda_2 = 1/3 \\ \lambda_3 = 0 \\ \lambda_4 = 0 \end{cases}$$



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(9)

Q 5 [20 pts]

a) [4 pts]

$$U = \{ Bx = 0 \text{ et } Cx \geq 0 \}$$

$x, y \in U$ $\Leftarrow$ $Bx = 0$ $Cx \geq 0$

$\lambda > 0$ $By = 0$ $Cy \geq 0$

✓ ⇓ ✓ ⇓ ✓

$B(x+y) = 0$ $C(x+y) \geq 0$

$B(\lambda x) = 0$ $C(\lambda x) \geq \lambda \cdot 0 = 0$

$\Rightarrow$ $x + y \in U$ ✓

$\lambda x \in U$

b) [4 pts]

Soit $x \in U$ $\quad (B^t \lambda_1 + C^t \lambda_2 , x)$

$= (B^t \lambda_1, x) + (C^t \lambda_2, x)$

$= (\lambda_1, \underbrace{Bx}_{0}) + (\underbrace{\lambda_2}_{\geq 0}, \underbrace{Cx}_{\geq 0}) \geq 0 \quad \forall x \in U$

$\Rightarrow B^t \lambda_1 + C^t \lambda_2 \in U^*$

c) [4 pts]

Soit $z \in (\operatorname{Im} B^t + C^t K_+)^*$

$\Leftarrow (z, B^t \lambda_1 + C^t \lambda_2) \geq 0 \quad \forall \lambda_1 \text{ et } \forall \lambda_2 \geq 0$

$\Leftarrow (Bz, \lambda_1) + (Cz, \lambda_2) \geq 0$

on prend $\lambda_2 = 0$ $\quad \forall \lambda_1 \; Bz = 0$ on prend $\lambda_1 = 0$ $\forall \lambda_2 \geq 0$

$\Rightarrow Cz \geq 0$

$\therefore z \in U$

Q5. (suite) [20 pts]

d) [4 pts]

On a que $\left(\operatorname{Im} B^* + C^* K_+\right)^* \subset U$

$\Rightarrow U^* \subset \left(\operatorname{Im} B^* + C^* K_+\right)^{**}$

$\Rightarrow U^* \subset \operatorname{Im} B^* + C^* K_+$

Mais $\operatorname{Im} B^* + C^* K_+ \subset U^*$

$\therefore =$

e) [4 pts]

La condition d'optimalité s'écrit

$x_0 \in U \quad \left(\nabla f(x_0), x - x_0\right) \geq 0 \quad \forall x \in U$

car U = convexe et cône

$\Rightarrow \left(\nabla f(x_0), y\right) \geq 0 \quad \forall y \in U$

car U est un cône

$\Rightarrow \nabla f(x_0) \in U^*$

$\Rightarrow \exists \lambda_1$ et $\lambda_2 \geq 0$ t.q.

$\nabla f(x_0) = B^* \lambda_1 + C^* \lambda_2$

$\lambda_1, \lambda_2 \geq 0$

et $B x_0 = 0$

$C x_0 \geq 0$

et $\left(\lambda_2, C x_0\right) = 0 \longrightarrow$ car $\left(\nabla f(x_0), x_0\right) = 0$ cond. de cône

$= \underbrace{\left(B^t \lambda_1, x_0\right)}_{0} + \left(C^t \lambda_2, x_0\right) = 0$