Mat-17441 optimisation   Automne 2008

Examen 1, solutionnaire

1. [20 pts]

$$f(x) = \frac{1}{2}(Ax, x) + \frac{1}{2}\|Bx\|^2 - (b, x) + c$$

$A \in M_{n \times n}$

$B \in M_{m \times n}$

a) [4 pts]

$$\nabla f = Ax - b + B^t B x \quad 2/$$

car $f(x) = \frac{1}{2}(Ax, x) + \frac{1}{2}(Bx, Bx) - (b, x) + c$

$2/$

$$f(x) = \frac{1}{2}(\underbrace{A + B^t B}_{\text{sym.}}\, x, x) - (b, x) + c$$

b) [4 pts]

$$H = A + B^t B \quad \text{sym.} \quad 4/$$

c) [4 pts]

$$H \geq 0 \quad 1/ \quad \text{car } (Hx, x) = (Ax, x) + (B^t Bx, x) \quad 2/$$

$$= (Ax, x) + \|Bx\|^2 \geq (Ax, x) \geq 0 \quad \forall x \neq 0$$

$\Rightarrow A + B^t B$ est sym et ≥ 0 $1/$

d) [4 pts]

1/ $\nabla f(x) = 0$

$\Leftrightarrow$ 3/ $(A + B^t B) x = b$

e) [4 pts] $\min_{x \in U} f(x)$

1/ U est fermé et f est coercive, strictement convexe

2/ f est coercive car f est quadratique

$f(x) = \frac{1}{2} (Cx, x) + \dots$

$C > 0$

$C = A + B^t B$

3/ ~~on~~ ~~$f(x)$~~

$\Rightarrow$ admet une sol (car f coercive)

unique (car f stric. convexe)

2. [20 pts]

a) [5 pts] $f(x,y) = x^2 + y^2 - xy + \sin x \sin y$

il suffit de montrer que f est coercive. 1/

$f(x,y) = g(x,y) + h(x,y)$

où $g(x,y) = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} > 0 \Rightarrow g$ coercive 2/

et $h(x,y) = \sin x \sin y \Rightarrow |h(x,y)| \leq 1$ 2/

$\Rightarrow f(x,y) = g + h \geq g - 1 \to \infty$
si $\|(x,y)\| \to \infty$

b) [5 pts]

$$f(x,y) = \frac{x^4}{12} + \frac{y^4}{12} + \frac{x^2}{2} + \frac{y^2}{2} + xy$$

$$f_x = \frac{x^3}{3} + x + y$$

$$f_y = \frac{y^3}{3} + y + x$$

2/ calcul de H

$f_{xx} = x^2 + 1$

$f_{xy} = 1 \qquad \Rightarrow H = \begin{pmatrix} x^2+1 & 1 \\ 1 & y^2+1 \end{pmatrix} \geq 0$ 1/

$f_{yy} = y^2 + 1$

car $(x^2+1)(y^2+1) - 1 \geq 0$ 2/

$x^2y^2 + x^2 + y^2 \geq 0$

Hilroy

e) [10 pts]

$f(x,y) = x^2 + 4y^2 + 2axy + 2x + 6y + 1$

$f(x,y) = (A\vec{x}, \vec{x}) + (\vec{b}, \vec{x}) + c$

avec $A = \begin{pmatrix} 2 & 2a \\ 2a & 8 \end{pmatrix}$ et $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

2/

Cas 1 :

Si $A > 0 \Rightarrow f$ admet une sol. unique

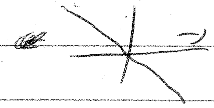
4/ $4(4-a^2)$

$4 - a^2 > 0 \Rightarrow \boxed{-2 < a < 2} \Rightarrow \boxed{\text{Min}}$

Cas 2 : $a = \pm 2$

2/ $a = 2$ $\quad f(x,y) = (x + 2y)^2 + 2x + 6y + 1$

$y = -\frac{x}{2}$ $\quad$ Si $y = -\frac{x}{2} \Rightarrow f(x,y) = 0 + 2x - 3x + 1$



$= 1 - x \to -\infty$

Si $x \to +\infty \to y \to -\infty$

$\|(x,y)\| \to \infty$

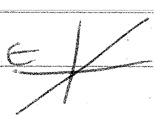
$\Rightarrow$ pas de minimum

2	4	2
2	4	3

$a = -2$ $\quad f(x,y) = (x - 2y)^2 + 2x + 6y + 1$

$y = \frac{x}{2} \Rightarrow f(x,y) = 0 + 2x + 3x + 1 = 5x + 1$

$\to -\infty$



Si $x \to -\infty \Rightarrow y \to -\infty$

$\|(x,y)\| \to \infty \Rightarrow$ pas de minimum

Cas 3 $\quad$ Les Pt. critiques sont des pts de selle

$\mathbb{R} \setminus [-2, 2]$ $\quad x + ay$

2/

3. [30 pts]

$$f(x,y) = \frac{x^2}{2} + x\cos y$$

a) [5 pts]

$f_x = x + \cos y = 0$ 1/

$f_y = -x\sin y = 0 \Rightarrow x = 0$ ou $\sin y = 0$

i) $x = 0 \Rightarrow \cos y = 0 \Rightarrow y = \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z}$

$\boxed{\left(0, \frac{\pi}{2} + k\pi\right)} \quad k \in \mathbb{Z}$ 2/

ii) $\sin y = 0 \Rightarrow y = k\pi \quad k \in \mathbb{Z} \Rightarrow \cos y = (-1)^k$

$(1, \pi(2k+1))$

$(-1, \pi 2k)$

ou

$\boxed{\left((-1)^{k+1}, k\pi\right)}$ 2/

$\left(-\cos\left(\frac{k\pi}{2}\right), \frac{k\pi}{2}\right) \quad k \in \mathbb{Z}$

b) [5 pts]

$$H = \begin{pmatrix} 1 & -\sin y \\ -\sin y & -x\cos y \end{pmatrix}$$ 1/

i) $\left(0, \frac{\pi}{2} + k\pi\right) \Rightarrow \cos y = 0 \Rightarrow \begin{pmatrix} 1 & -\sin y \\ -\sin y & 0 \end{pmatrix}$

$\Rightarrow \det H < 0$

$\Rightarrow H$ indéfinie $\Rightarrow$ $\boxed{\text{pt-selle}}$ 2/

ii) $\left((-1)^{k+1}, k\pi\right) \quad \sin y = 0 \Rightarrow x = -\cos y$

$x^2 = -x\cos y$

$$\Rightarrow H = \begin{pmatrix} 1 & 0 \\ 0 & -x\cos y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$H > 0 \Rightarrow$ min. 2/

$(-\cos k\pi, k\pi)$

c) [5 pts]

Non! car $(\overset{(x,y)}{0}, k\pi)$ $\|(x,y)\| \to \infty$

$k = 1,2,3\ldots$

2/

car $(0, y) \to \infty$

$f(0, k\pi) = 0 \times 0 = 0 \not\to \infty$

$f(0, y) = 0$

3/

d) [8 pts]

$$\begin{pmatrix} 1 & -\sin y_k \\ -\sin y_k & -x_k \cos y_k \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -x_k - \cos y_k \\ x_k \sin y_k \end{pmatrix}$$

2/

$x_0 = 1$ et $y_0 = \frac{\pi}{2}$

$$\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$ 2/

$$\Rightarrow \begin{cases} \Delta x = -1 \\ \Delta y = 0 \end{cases}$$ 2/

$\Rightarrow$ $x_1 = x_0 + \Delta x = 1 + (-1) = 0$ 2/

$y_1 = y_0 + \Delta y = \frac{\pi}{2} + 0 = \frac{\pi}{2}$

e) [7 pts]

$$\vec{x}_{k+1} = \vec{x}_k - \rho \vec{\nabla} f(\vec{x}_k)$$

$$\Leftrightarrow \begin{cases} x_{k+1} = x_k - \rho\,(x_k + \cos y_k) \\ y_{k+1} = y_k - \rho\,(-x_k \sin y_k) \end{cases}$$ 2/

$x_0 = 1$, $y_0 = \frac{\pi}{2}$ et $\rho = 1/10$

$$\begin{cases} x_1 = 1 - \frac{1}{10}\left(1 + \cos\frac{\pi}{2}\right) \\ y_1 = \frac{\pi}{2} - \frac{1}{10}\left(-\sin\frac{\pi}{2}\right) \end{cases}$$ 2/

$\Leftrightarrow \quad x_1 = 1 - \frac{1}{10}(1+0) = 9/10 = .9$

3/

$y_1 = \frac{\pi}{2} - \frac{1}{10}(-1) = \frac{\pi}{2} + \frac{1}{10} = 1.6$

Hilroy

4. [15 pts]

$$\begin{cases} 6x + y + \cos x = 0 \\ x + 2y + \sin y = 0 \end{cases}$$

$f_x = 6x + y + \cos x \Rightarrow f = 3x^2 + xy + \sin x + g(y)$

$f_y = x + 2y + \sin y \quad f_y = x + g'(y)$

5/ pour f $\Rightarrow g(y) = y^2 - \cos y$

$f(x,y) = 3x^2 + y^2 + xy + \sin x - \cos y$

Montrons que f est coercive et strictement convexe.

i) $f = g + h$

où $g = 3x^2 + y^2 + xy = (x \; y) \underbrace{\begin{pmatrix} 3 & 1/2 \\ 1/2 & 1 \end{pmatrix}}_{>0} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow g$ coercive

5/ h est borné $|h| \leq 2$

$\Rightarrow f$ coercive.

ii) $f_x = 6x + y + \cos x$

$f_y = x + 2y + \sin y \qquad \Rightarrow f$ strictement convexe

5/ $f_{xx} = 6 - \sin x$

$f_{xy} = 1 \qquad \Rightarrow \begin{pmatrix} 6 - \sin x & 1 \\ 1 & 2 + \cos y \end{pmatrix}$

$f_{yy} = 2 + \cos y$

Hilroy

Mais $\quad 6 - \sin x \geq 5 \quad$ car $1 \geq \sin x \quad \forall (x,y)$

et $\quad (6-\sin x)(2+\cos y) - 1$

$= 12 - 2\sin x + 6\cos y - 1$

$\geq 11 - 2 - 6 = 3 > 0 \quad \forall x,y$

$\Rightarrow \quad H > 0 \quad \Rightarrow \quad f$ strict. convexe.

So $\quad \min f(x,y)$ admet une sol. unique $(\bar{x},\bar{y})$

$\Leftarrow \quad Df((\bar{x},\bar{y}))=0$

5. [15 pts]

a) [7 pts]

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$$

$$= \lambda_1 x_1 + (1-\lambda_1)\left[\frac{\lambda_2 x_2}{1-\lambda_1} + \frac{\lambda_3}{1-\lambda_1} x_3\right]$$

et $\frac{\lambda_2}{1-\lambda_1} + \frac{\lambda_3}{1-\lambda_1} = 1$ 2/

$\therefore$ $f(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)$ 2/ car f convexe

$$\le \lambda_1 f(x_1) + (1-\lambda_1) f\left(\frac{\lambda_2}{1-\lambda_1} x_2 + \frac{\lambda_3}{1-\lambda_1} x_3\right)$$

2/ car f convexe

$$\le \lambda_1 f(x_1) + (1-\lambda_1)\left[\frac{\lambda_2}{1-\lambda_1} f(x_2) + \frac{\lambda_3}{1-\lambda_1} f(x_3)\right]$$

1/

$$\le \lambda_1 f(x_1) + \lambda_2 f(x_2) + \lambda_3 f(x_3)$$

b) [8 pts]

$$f(x) = e^x \text{ est convexe.}$$

$$a^p = e^{p \ln a}$$

On pose: $\lambda_1 = \frac{1}{p}$

$\lambda_2 = \frac{1}{q}$ $\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 1$

$\lambda_3 = \frac{1}{r}$

$x_1 = p \ln a$

$x_2 = q \ln b$

$x_3 = r \ln c$

$$f(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \lambda_3 f(x_3)$$

$$f(\ln a + \ln b + \ln c) \leq \frac{e^{p \ln a}}{p} + \frac{e^{q \ln b}}{q} + \frac{e^{r \ln c}}{r}$$

$$e^{\ln(abc)} = abc \leq \frac{a^p}{p} + \frac{b^q}{q} + \frac{c^r}{r}$$