Théorie formelle des nombres

Chap. 3 : Formal number theory
[Elliott Mendelson, *Introduction to Mathematical Logic* (3e édition)]

Système **S** (cf système $\mathcal{N}$ de Hamilton, p. 116).
On désigne par « ' » la fonction *successeur* (i.e. t' représente t+1).

AXIOMES

(réécriture)

(S1′) $t = r \Rightarrow (t = s \Rightarrow r = s)$
(S2′) $t = r \Rightarrow t' = r'$
(S3′) $0 \neq t'$
(S4′) $t' = r' \Rightarrow t = r$
(S5′) $t + 0 = t$
(S6′) $t + r' = (t + r)'$
(S7′) $t \cdot 0 = 0$
(S8′) $t \cdot r' = (t \cdot r) + t$

(plus un axiome d'induction)

Exemples de dérivation formelle

PROPOSITION 3.2 For any terms t, r, s, the following wfs are theorems of S.

(a) $t = t$
(b) $t = r \Rightarrow r = t$
(c) $t = r \Rightarrow (r = s \Rightarrow t = s)$
(d) $r = t \Rightarrow (s = t \Rightarrow r = s)$
(e) $t = r \Rightarrow t + s = r + s$
(f) $t = 0 + t$
(g) $t' + r = (t + r)'$
(h) $t + r = r + t$
(i) $t = r \Rightarrow s + t = s + r$
(j) $(t + r) + s = t + (r + s)$
(k) $t = r \Rightarrow t \cdot s = r \cdot s$
(l) $0 \cdot t = 0$
(m) $t' \cdot r = t \cdot r + r$
(n) $t \cdot r = r \cdot t$
(o) $t = r \Rightarrow s \cdot t = s \cdot r$

Proof

(a) 1. $t + 0 = t$
2. $(t + 0 = t) \Rightarrow (t + 0 = t \Rightarrow t = t)$
3. $t + 0 = t \Rightarrow t = t$
4. $t = t$

(b) 1. $t = r \Rightarrow (t = t \Rightarrow r = t)$
2. $t = t \Rightarrow (t = r \Rightarrow r = t)$
3. $t = r \Rightarrow r = t$

(c) 1. $r = t \Rightarrow (r = s \Rightarrow t = s)$
2. $t = r \Rightarrow r = t$
3. $t = r \Rightarrow (r = s \Rightarrow t = s)$

(d) 1. $r = t \Rightarrow (t = s \Rightarrow r = s)$
2. $t = s \Rightarrow (r = t \Rightarrow r = s)$
3. $s = t \Rightarrow t = s$
4. $s = t \Rightarrow (r = t \Rightarrow r = s)$
5. $r = t \Rightarrow (s = t \Rightarrow r = s)$