Structure in Mathematics

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This article is intended to describe what mathematicians would commonly call a 'mathematical structure'. Such a structure is essentially a list of mathematical operations and relations and their required properties, commonly given as axioms, and often so formulated as to be properties shared by a number of possibly quite different specific mathematical objects. To make this more explicit, I formulate first three familiar examples of such structures: a group, a metric space, and a topological space.

A group G is a set in which any two elements x and y have a specific product xy, an element of G, provided these products enjoy the following properties. First, the product is associative; for any three elements x, y, and z of G one has

$$(xy)z = x(yz).$$

Second, the group G has a 'unit' or 'identity' element 1 so that always

$$x1 = x = 1x.$$

Finally, to each element x there is in G an 'inverse' element x^{-1} such that always

$$xx^{-1} = 1 = x^{-1}x.$$

For example, the positive real numbers form a group under the usual operation of multiplication, as do the four (complex) numbers $1, i, -1, -i$. The integers $m, n, k \ldots$ form a group when the addition of $m + n$ of integers is taken to be the group multiplication. There are many other—and varied—examples of groups. Thus the eight symmetries of a square (the operations of rotation or reflection) form a group when the multiplication is taken to be the composition (x followed by y). There are many other important symmetry groups, for instance, those arising in crystallography and in physics. The 'representation' of an abstract group by geometric transformations has proved to be of capital importance. These are only some of the reasons for the utility of groups in mathematics, and hence for the importance of this 'structure'.

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One often uses the phrase 'the structure of ... '. Thus a mathematical object M is said to have the 'structure of a group' when there is given an operation of multiplication on the elements of M which satisfies the group axioms.

A metric space M is a set together with a 'distance' function d which assigns to any two points s and t of M a non-negative real number $d(s,t)$ called the 'distance' from s to t. This distance must have three properties: $d(s,t) = 0$ if and only if $s = t$; $d(s,t) = d(t,s)$ for all s and t, and, for any three points r, s, and t,

$$d(r,t) \leq d(r,s) + d(s,t).$$

This last requirement is commonly called the 'triangle' axiom. The ordinary plane or the 3-dimensional Euclidean space with the familiar measure of distance between its points is a metric space. There are other, sometimes bizarre, metric spaces and there can be much bigger such spaces—for instance, the space of all functions of some given type. One says that a mathematical object M 'has a metric structure' when any such a distance function d is given for the points of M. A given set might well have several different metric structures.

In any metric space M the 'open disc' about a point s of M with radius a positive real number x is defined to be the set U of all those points t of M for which $d(t,s) < x$. Such a disc is also called a 'neighborhood' of s. Any union of open discs of M is called an 'open set' of M; thus 'open' in M means roughly a set with no 'boundary' points. For many geometric purposes the open sets are more important than the actual numerical distances. For example, they can be used to define the 'continuity' of functions on M to a second metric space N. Intuitively, a function f from M to a second space N is continuous when 'nearby' points go to nearby ones. Specifically, if f carries the point s of M to a point w of N, then to each open disc D about w there must be a disc D' about s such that f carries D' inside D.

These observations lead to the general notion of a 'topology'. A topological space X is a set together with a specification that certain subsets of X are to be regarded as 'open'. This specification must satisfy several axioms: The whole space X and its empty subset $\emptyset$ are both open, while the intersection (common part) of any two open subsets of X is open, as is the union of any collection of open sets. Any metric space M, with open sets defined in terms of open discs as described above, is a topological space in this sense. One then says that any metric space has the 'structure' of a topological space. But not every topological space is a metric one; a famous mathematical problem, now solved, has been that of determining which topological spaces 'are' metric (*i.e.*, can be given a metric with the same open sets).

A group is one (among many) examples of an 'algebraic' structure, while a metric or a topology is an example of a 'topological' structure. There are many more elaborate types of such 'structures'. One suggestive combined example is that of a topological group. This is a set G which has both a group structure and a topological structure, in such a way that the group multiplication and the group inverse are both continuous in the sense of the given topology.

For example, the ordinary unit circle S^1 (all points x, y in the plane, with $x^2 + y^2 = 1$) is a topological group. Here, the topology is given by the usual distance between points in the plane. Here, the group multiplication of two rotations is given by composing rotations, where each point (x, y) on the circle S^1 is viewed as that rotation of the circle required to carry the point $(1, 0)$ to the point (x, y). In this case, one commonly says that the unit circle has the structure of a topological space as well as that of a group. Such uses of the word 'structure' are typical for mathematics. Briefly, a mathematical object 'has' a particular structure when specified aspects of the object satisfy the (standard) list of axioms for that structure. The 'aspects' here in question can be selected subsets, specific operations, or other composite features.

This notion of 'structure' is clearly an outgrowth of the widespread use of the axiomatic method in mathematics. This method was initially deployed primarily to give a rigorous description (called an 'axiomatization') of some unique mathematical object. Euclid's original axioms for geometry had subtle imperfections, but David Hilbert's famous 1899 book *Grundlagen der Geometrie* gave a definitive and complete set of axioms for the points and lines in the usual euclidean plane. These axioms were 'categorical', in the sense that they had, up to isomorphism, only one model—'the' euclidean plane. Similarly, the real numbers can be described categorically by axioms (as a complete ordered field). (Here I am ignoring the possible existence of 'non-standard' models.) Then one may say that these axioms describe 'the' structure of the system of real numbers.

However, the axiomatic method applies not just to provide such categorical descriptions of mathematical objects, but also to describe the common properties of large classes of mathematical objects—such as the class of all groups or that of all topological spaces. In particular, the axioms for a group were apparently first formulated by Arthur Cayley in 1852. He (and everyone else at that time) appears to have forgotten this paper; he stated the axioms again in a paper written in 1882 for the *American Journal of Mathematics.* The group axioms were also formulated (independently) by several other mathematicians, including Leopold Kronecker in 1870 and Heinrich Weber in 1882. Axioms for the real numbers were also formulated by several mathematicians—K. Weierstrass (1870), C. Meray (1879), G. Cantor, and R. Dedekind (1872). In these cases, the need for general axiomatic methods was apparent well before Hilbert.

In the 20th century, axiomatic methods were developed vigorously in mathematics. In algebra, the school of Emmy Noether (1916–1933) in Göttingen was especially influential. Her development of 'modern algebra' emphasized various algebraic structures and their homomorphisms—especially the notion of a 'ring'. Here a ring is a set of elements with two suitable (binary) operations, those of addition and of multiplication, satisfying axioms—chosen so as to include rings of numbers (in particular, rings of algebraic integers) and rings of functions (in particular, rings of polynomials). The intent was that the axioms would codify and simplify the properties of polynomials as they were used in algebraic geometry and elsewhere. Noether also emphasized the point that a clearer axiomatic formulation would lead to better understanding—notably in the case of the Galois theory, where the older typical expositions were incredibly obscure.

Linear algebra is an example of the clarity possible with a emphasis on structure. A vector was initially considered as a string of numbers (its coordinates), and vectors were transformed by multiplication by a square matrix of numbers. In geometry and in tensor analysis this led to a proliferation of indices. The suitable axioms were those for a vector space—axioms on addition of vectors and multiplication of a vector by a scalar. This approach, previously neglected, was clearly emphasized in a 1917 book by Herman Weyl on relativity theory. The axioms for a topological space were first formulated (independently) in 1913 by Weyl (in Göttingen) and by Felix Hausdorff (in Bonn) (Incidentally, Hausdorff, under the name Paul Mongé, doubled as a poet and philosopher of Nietschean type). In analysis, Hilbert and others introduced extensive discussions of topological spaces whose 'points' are actually functions. These spaces were then axiomatized as Banach spaces and as Hilbert spaces. They were used not only in classical mathematical problems, but also notably in quantum mechanics. In this and in various other aspects many familiar mathematical problems were formulated in terms of an axiomatized 'structure', and the study of such structure provides powerful tools.

However, it never was the case that all of mathematics was concerned with such structure. Important classical questions about analytic functions of a complex variable remained quite concrete, *i.e.*, specific, while standard questions about numbers (why are π and e transcendental; how are the prime numbers distributed?) are hardly structural. Much of the active study of partial differential equations and their solutions (*e.g.*, for fluid mechanics) cannot naturally be described in terms of axiomatized 'structures'. Number theory remains quite concrete. For these and related reasons, we cannot claim that mathematics is just the study of axiomatically defined structures—but we can observe that such structures are widely and effectively used.

In the early 20th century the word 'structure' was also used with a different mathematical meaning. This was the sense in which the 'structure' of a mathematical object is described by the way in which such an object could be reconstructed from simpler objects of the same type. I shall next give some examples of this usage.

An abelian group A is one in which the operation of multiplication is commutative ($xy = yx$ for all x and y in A). The simplest abelian groups are the 'cyclic' ones, consisting of all the powers of a single element (for example, all the six rotations of a regular hexagon). Given two abelian groups A and B one can form another group $A \times B$ called their 'product', and consisting of all pairs (a, b) of elements a from A, b from B, suitably multiplied. The so-called 'structure' theorem for finite abelian groups A states that every such A can be represented as an iterated product of cyclic groups—and that this representation is essentially unique. The relevance of this theorem comes because it is known to arise in several different contexts—in Galois theory, in number theory and in the homology groups used in topology. It is a powerful and unifying 'structure theorem'.

Another example of such theorems arises from the study of possible number systems: the real numbers, the complex numbers, the quaternions (where the multiplication is not commutative) and matrix algebras. All these can be described axiomatically as 'linear associative algebras'. There are some penetrating theorems by Wedderburn as to how all finite dimensional simple such algebras can be built up from pieces which are matrix algebras or division algebras (like the quaternions). These results are the 'structure theorems' for algebras. They feature the alternative meaning of 'structure': The reconstruction of mathematical objects by suitable combination of their simplest examples.

The word 'structure' was also used in various other informal ways in the school of modern algebra led by Noether in Göttingen. For example, both Oystein Ore (later professor at Yale) and I had studied in Göttingen, and both of us had talked cheerfully about 'structure'. Thus in the early 1930's algebraists studied the array L of all the subgroups of a group (or of all the subrings of a ring). The form of this array of subobjects revealed something about the 'structure' of that group or that ring. Thus two elements of L have both an intersection (common part) and a union, and these two operations of union and intersection can be suitably axiomatized. The resulting type of axiomatized object was called a 'lattice' by Garrett Birkhoff and a 'structure' by Oystein Ore. Evidently, Ore's idea was that this object L pictured the real 'structure' of the group or ring in question. Subsequently the word 'structure' in this axiomatic sense has been dropped, in favor of the word 'lattice'—but a lattice of subobjects is still regarded as a description of structure.

Here is another example. In 1933, just returned from my studies in Göttingen, I published in the *Bulletin of the American Mathematical Society* (March 1934) an abstract (no. 87) 'General properties of algebraic systems'. There I stated (without proof) the following theorem: 'Any two isomorphic systems have the same structure'. To give a proof of such a theorem, I must have had some specific definition of 'structure'. I no longer recall that definition. But it is clear that algebraists in 1933 were happy to think about structure, with quite various meanings.

By now, the word 'structure' usually has the meaning we first presented: it refers to a class of mathematical objects described by axioms. The current wide prevalence of this usage is probably due to the influence of Nicholas Bourbaki. This is the pen name of a well known group of (once young) French mathematicians, who started work in the middle 1930's. Their massive and widely used multivolume treatment of the *Éléments de Mathématique*, with a first part entitled 'Les structures fondamentales de l'analyse' began with volume 1, 'Théorie des ensembles, Fascicule de résultats' (Paris: Hermann, 1939). In this volume, Bourbaki carefully describes what he means by a structure of some specific type T. We do not really need to use this description, but we will now present it, chiefly to show both that one can indeed define 'structure' and that the explicit definition does not really matter. It uses three familiar operations on sets: the product $E \times F$ of two sets E and F, consisting of all the ordered pairs (e, f) of their elements, the power set $P(E)$, consisting of all the subsets S of E, and the function set E^F consisting of all the functions mapping F into E. For example, a topological structure on E is given by an element T of $P(P(E))$ satisfying suitable axioms—it is just the set consisting of all open sets, that is, the set T of all those sets U in $P(E)$ which are open in the intended topology. Similarly, a group structure on a set G can be viewed as an element $M \in G^{G\times G}$ with the usual properties of a group multiplication. With these examples in mind, one may arrive at Bourbaki's definition of a structure, say one built on three given sets E, F, and G. Adjoin to these sets any product set such as $E \times F$, any function set such as E^G and any power set such as $P(G)$. Continue to iterate this process to get the whole scale (*échelle*) of sets M successively so built up from E, F, and G. On one or more of the resulting sets M impose specified axioms on a relevant element (or elements) m. These axioms then define what Bourbaki calls a 'type' T of 'structure' on the given sets. This clearly includes algebraic structures like groups, topological structures, and combined cases such as topological groups (the definition also includes many bizarre examples of no known utility).

Actually, I have here modified Bourbaki's account in an inconsequential way; he did not use function sets such as E^F. This modification does not matter; as best I can determine, he never really made actual use of his definition, and I will not make any use here of my variant. It is here only to show that it is indeed possible to define precisely 'type of structure' in a way that covers all the common examples.

Bourbaki's subsequent (largely post-war) massive and enthusiastic treatise covered many branches of mathematics and many useful types of structures. It had a wide influence. In particular, it introduced much new terminology; it was undoubtably largely responsible for the present extensive use in mathematics of the term 'structure'. But Bourbaki tended to include chiefly those parts of mathematics which are best described by axioms for one of his 'structures'. The treatise included topics of algebra and of topology and of general or abstract analysis—but there is no Bourbaki volume on the classically dominant subject of the functions of one complex variable. Applied mathematicians complained from the beginning that Bourbaki's structures did not cover their interests in differential equations and more recently work in topics such as chaos, fractals and dynamical systems.

But for many other mathematical purposes, the notion of different types of mathematical structure does provide an effective means of organizing and of understanding mathematics.

For example, structures lead to 'morphisms'; for each type T of structure there is a consequent notion of a homomorphism—in brief, a 'morphism'—from one model of the structure to another. Emmy Noether had emphasized the importance of this notion for a homomorphism $f : G \to H$ from a group G to a group H (in her case, always 'onto' H). It is a function f which assigns to each element x of G an 'image' element $f(x)$ in H in such a way that always

$$f(xy) = f(x)f(y).$$

In other words, the 'structure' given by the group multiplication is preserved; products map to products (and in consequence, inverses map to inverses). For topological spaces X and Y a morphism $f : X \to Y$ is a function from the set X to the set Y such that the 'inverse image' f^{-1} preserves open sets; that is, such that for each set U open in Y its inverse image $f^{-1}U$ (the set of all x in X with $f(u)$ in U) is also an open set in X. This definition of morphism of course includes the classical notion of a continuous function of a real or a complex variable. In this instance, the use of the inverse image means that the power set $P(Y)$ is regarded as a 'contravariant' functor; for some other structures, the power set may be covariant, *i.e.*, one using the direct image of a subset.

For any type T of structure the homomorphisms compose, given morphisms $f : G \to H$ and $k : H \to L$ of models the composite $k \circ f$ (that is, f followed by k) is a morphism $k \circ f : G \to L$. This composition of morphisms is associative, in that the equation

$$m \circ (k \circ f) = (m \circ k) \circ f$$

holds and has for each model G an identity morphism $G \to G$, which acts as an identity under any composition. These are the properties involved in the definition of a category; thus it turns out that all the models of a given type T of structure together with their morphisms form a category. Here a category, consisting of objects (models) and arrows (morphisms), is described as in the article by S. Awodey to be published in the next issue of this journal.

Formally this result (that all models of a type of structure form a category) could be explicitly proved from the Bourbaki definition of a structure, as given above and with attention to the variance of the power set construction. Curiously, this proof has never (to my knowledge) been actually written out; to do so would be just a mechanical exercise. Bourbaki never did this, perhaps because he clearly did not like categories. Eilenberg–Mac Lane (who first developed Category Theory) never did this because they found it needless and considered the Bourbaki definition of 'type of structure' in an '*échelle*' of sets a cumbersome piece of pedantry. At one time (in 1954), I tried to argue this general issue at a meeting of Bourbaki: I failed to convince him.

But the notion of 'morphism' matters. Bourbaki and many others always put a central question to any new mathematical gadget: 'What are the morphisms?' Structure inevitably leads to morphisms for Bourbaki and everyone else.

For example, once vector spaces are defined axiomatically, it becomes inevitable to ask for the morphisms $V \to W$ of such spaces. These are exactly the linear transformations (those functions from V to W which preserve vector addition and multiplication of vectors by scalars). The given transformation can then be represented (relative to bases in the spaces V and W) by matrices. It is this observation which has greatly clarified the earlier poorly motivated calculations with matrices: they are representations of morphisms. This is a good illustration of the sense in which ideas about structure clarify understanding.

In mathematics, there is thus a common and effective notion of a mathematical structure, even though one did not usually bother to attend to a formal definition of 'structure'. However, an emphasis on structure does provide some guidance to the pursuit of mathematical research. One may search for new structures which might come to play a central role; there is for instance currently a proliferation of new structures (quantum groups, tensor categories, tangles, *etc.*) to be used in quantum field theory and/or in low-dimensional topology. Also one may search for other new combined structures, like the combination 'topological group'. And for some—by no means all—types of structures one can search for 'structure theorems' which decompose models of the intended structure into simpler components. The notion of a structure does thus provide a type of guide and explanation of some aspects of the direction in which mathematical research evolves. But what are the philosophical consequences?

The fundamental question may now be this: to what extent can the explicit mathematical notion of 'structure' serve as a support for an adequate 'structuralist' philosophy of mathematics? We can also ask how this view may compare with other philosophies, say with set-theoretical platonism, especially in the form which asserts that mathematics is founded on set theory, often with a platonist view of the existence of sets.

First, as just noted, the emphasis on 'structure' is close to the active process of some mathematical research; most other philosophical doctrines are not connected with the process of discovery. But such emphasis on the notion of structure does not suffice to explain why certain structures—for example, group structure—play such a central role in mathematics.

In the study of structure, there is really no such thing as a 'mathematical object', even though we have initially spoken of such objects in first describing what is meant by 'structure'. This is because the use of structure carries with it the observation that mathematical objects can be described only 'up to isomorphism'. Thus categorical axioms, say for the real numbers, do not really describe a unique model of the reals. We know only that any two standard models (say Dedekind cuts or equivalence classes of convergent sequences of rationals) must be isomorphic. In category theory, all the canonical constructions, as by adjoint functors, produce objects only 'up to isomorphisms'. For example, the product of two sets or two groups need not consist of the usual ordered pairs of elements.

This differs from a common set-theoretic view, where an ordered pair is described by a specific set, or where the ordinal numbers may be defined to be the von Neuman ordinals. Other models of ordered pairs or of ordinals would do as well.

All mathematics can indeed be built up within set theory, but the description of many mathematical objects as structures is much more illuminating then some explicit set-theoretic description.

An emphasis on structure makes it clear that the same type of mathematical structure can and does appear in wholly different contexts. For example, groups and rings appear all over. Moreover, the same model of a structure has many different physical realizations. All infinite cyclic groups are isomorphic, but this infinite cyclic group appears over and over again—in number theory, in ornaments, in crystallography, and in physics. Thus, the 'existence' of this group is really a many-splendored matter. An ontological analysis of things simply called 'mathematical objects' is likely to miss the real point of mathematical existence.

A famous query of Eugene Wigner asks for an explanation of the unreasonable effectiveness of mathematics—especially its effectiveness in the various sciences. The emphasis on structure does not really provide a response, except to say that different parts of science frequently exhibit some common structure. One might say that mathematics, too, is a science;

namely it is that science whose objects are the forms or structures which are common to different specific scientific realizations. I am inclined to restate this by asserting that mathematics is protean science; its subject matter consists of those structures which appear (unchanged) in different scientific contexts.

However, a doctrine of structuralism does not seem to settle all the relevant questions in the philosophy of mathematics. For example, there can be quite different view of structure–as something arising in set theory and then formulated in Bourbaki's typical structures, or as something located in some ethereal category. Talking of structures does not explain why certain structures (*e.g.*, groups or topological spaces) play such a dominant role. Structure theorems do not really exemplify the important role of calculations or of geometric intuition in the study of particular structures. And, as already observed, there are many parts of mathematics which do not easily fit in a structure—even though some of these parts (*e.g.*, differential equations) resemble structures in the sense that the same piece of mathematics may have varied interpretations in science. And finally, for this protean character of mathematical concepts, I do not have an adequate philosophical explanation. What is it about the nature of the world which makes it possible for the same simple structural form to have so many and such varied realizations? Does the prevalence of situations where so many diverse objects have the same mathematical structure come about from the nature of the world or from the character of our means of understanding that world? In short, why are groups, spaces and rings present in so many widely different realizations?

The emphasis on mathematical structure has served wonderfully to organize much of mathematics and to clarify some previously confused topics, such as Galois theory, matrix calculations, differential geometry, and algebraic topology. But 'structure' seems at best just one possible aspect of an adequate philosophy of mathematics. Such an adequate philosophy is not now available. Many philosophers have followed the model set by Wittgenstein—discussing questions of putative philosophical importance, but with little or no attention to the rich and varied actual aspects of mathematics.

Abstract. The article considers structuralism as a philosophy of mathematics, as based on the commonly accepted explicit mathematical concept of a structure. Such a structure consists of a set with specified functions and relations satisfying specified axioms, which describe the type of the structure. Examples of such structures such as groups and spaces, are described. The viewpoint is now dominant in organizing much of mathematics, but does not cover all mathematics, in particular most applications. It does not explain why certain structures are dominant, not why the same mathematical structure can have so many different and protean realizations. 'Structure' is just one part of the full situation, which must somehow connect the ideal structures with their varied examples.